Quantum field theory of self-interacting systems

1973 J. Phys. A: Math. Nucl. Gen. 6 1388

(http://iopscience.iop.org/0301-0015/6/9/014)

J. Phys. A: Math., Nucl. Gen., Vol. 6, September 1973. Printed in Great Britain. © 1973

Quantum field theory of self-interacting systems

P B Burt and J L Reid

Department of Physics and Astronomy, Clemson University, Clemson, SC 29631, USA

Received 16 January 1973, in final form 6 April 1973

Abstract. Two self-interacting quantum field theories arising from the interaction lagrangian $\lambda\phi^4$ are discussed. Proceeding from exact particular solutions of the field equations which reduce to the positive and negative frequency solutions of the free field theory for $\lambda = 0$, the theories are quantized either by requiring (*a*) the commutator of the positive and negative frequency solutions to be independent of λ at $t = 0$ or (*b*) the propagator formed from the operator solutions of the field equations to be invariant with respect to translations. The solutions satisfying (*a*) are appropriate for scattering problems extending from $t = 0$ to $t = \infty$ while solutions satisfying (*b*) are valid from $t = -\infty$ to $t = \infty$. The solutions satisfying (*a*) become independent of λ for $t = 0$ but continue to depend on λ for large times while those satisfying (*b*) contain λ for all times. Both types of solutions satisfy the interacting field equations for all times. For times other than zero in case (*a*) and for all times in case (*b*) the equal-time commutator of the positive and negative frequency fields contains λ. The propagator for case (*b*) is calculated and is found to have poles at $(2n+1)m$, where m is the mass of the associated free field and n is an integer.

1. Introduction

In the study of quantization of self-interacting systems one of the most important examples has been the scalar or pseudoscalar field with interaction lagrangian of the form $\lambda\phi^4$. Although the original stimulus for studying this type of system arose in connection with renormalization of the perturbation series for nucleons and mesons interacting by a Yukawa coupling (Matthews 1950), in recent years the theory, because of its relative simplicity, has been the subject of axiomatic and mathematical studies independent of perturbation theory (Glimm and Jaffe 1970, Segal 1960, 1964, Streater and Wightman 1964). In addition to these, the existence of resonances in multiple meson systems is sufficient reason for examining this type of theory.

The usual perturbation theory approach to this problem assumes the existence of in and out fields which are to describe the physical system initially and finally. These fields satisfy free field equations. This property is reasonable in a scattering theory where one subsystem (target) is isolated at the beginning and end of an experiment from the other subsystem (incident beam). It becomes more difficult to justify in a field theory where even the incident systems are sources of the field and can be expected to interact with this field. Of course, the problem of self-interactions in field theory is quite old and is a source of difficulties in both classical and quantum field theory. At present, as is well known, the most common approach to this problem is to adjust the interaction so that infinities in the self-energy arising in perturbation theory are cancelled term by term in perturbation theory by self-interactions. This point of view places the self-interaction on the same basis as the interaction of one subsystem with another—

specifically, the fields at $\pm\infty$ are assumed to satisfy free field equations and the self-interaction is included in the source (current) of the interacting field.

In this paper we will give some results of studying the quantization of self-interacting systems independent of the property described above. While free fields have a place in this study they are not limits of the interacting fields for large times. By this we mean that the fields always satisfy interacting field equations. This places the self-interaction on its most plausible basis. As the coupling constant vanishes (independent of time) the operators reduce to the free field operators. (To make a fair appraisal of the re-normalization point of view we would have to agree that allowing the self-interaction to change the bare mass of a particle into its physical mass also implies that the self-interaction is always present. However, in this type of theory it is necessary to assume that no subsequent inconsistencies develop in the detailed structure of the theory. It is well known that for higher spin systems this assumption is invalid.)

To summarize, we study self-interactions by solving the differential equations of the interacting fields and then quantizing the solutions.

The outline of this paper is as follows. We will consider throughout the case of a neutral scalar field with self-interaction $\lambda\phi^3$ (in the field equation). In § 2 one-dimensional solutions of the field equations will be quantized, using commutation relations similar to the equal-time commutation relations of linear field theories. The resulting operators reduce to free field operators as λ approaches zero (and also at $t = 0$). The propagator is not invariant with respect to translations and the field operators contain λ at $t = \pm\infty$. In § 3 an alternative method of quantization is discussed in 3+1 dimensions. The propagator in this case is invariant with respect to translations. It also has poles at $(2n+1)m$, where m is the mass appearing in the field equation and n is an integer. In the last section we summarize the results. In appendix 1 details of the solution of the field equations are given while the convergence of a propagator is considered in appendix 2.

2. Commutators specified at $t = 0$

The field equation considered here is the well known generalization of the Klein–Gordon equation,

$$(\partial_\mu\partial^\mu + m^2)\phi + \lambda\phi^3 = 0, \tag{1}$$

where m is the mass of the particle and λ is the coupling constant. We have discussed particular solutions of this equation suitable for quantization elsewhere (Burt and Reid 1972). In this section we will consider the one-dimensional theory given previously. In particular we will show that the quantized solutions give rise to propagators which are not invariant under translations and are thus not suitable for describing scattering problems extending from $t = \pm\infty$. The solutions are

$$\phi^{(\pm)}(t) = A_\lambda^{(\pm)}\,\mathrm{e}^{\mp imt}(1 + c\lambda A_\lambda^{(\pm)2}\,\mathrm{e}^{\mp 2imt})^{-1} \tag{2}$$

with

$$c = \frac{1}{8m^2}. \tag{3}$$

As is evident from inspection these solutions contain the coupling constant for large t. For zero interaction they reduce to solutions of the free field equations. Finally, each solution is constructed from positive or negative frequency solutions.

As a first method of quantization we require

$$[\phi_\lambda^{(+)}(0), \phi_\lambda^{(-)}(0)] = 1. \tag{4}$$

This requirement is given in analogy with free field theory and insures that the commutator will be identical with the free commutator for zero interaction (to within a constant).

Specifying the commutator enables us to find the coefficients $A_\lambda^{(\pm)}$ in terms of free field annihilation and creation operators. One solution is (Burt and Reid 1972)

$$A_\lambda^{(+)} = \sum_{n=0}^{\infty} (c\lambda)^n \alpha_n a^{2n+1} \tag{5}$$

where α_n are numerical coefficients. A similar result is obtained for $A^{(-)}$ with a replaced by $a^\dagger$. The commutator of a and $a^\dagger$ is unity.

Now, we construct the propagator, again by analogy with free field theory,

$$\begin{aligned} P(t, t') &= \langle 0|\phi_\lambda^{(+)}(t)\phi_\lambda^{(-)}(t')|0\rangle\theta(t-t') + \langle 0|\phi_\lambda^{(+)}(t')\phi_\lambda^{(-)}(t)|0\rangle\theta(t'-t) \\ &= \langle 0|[\phi_\lambda^{(+)}(t), \phi_\lambda^{(-)}(t')]|0\rangle\theta(t-t') + \langle 0|[\phi_\lambda^{(+)}(t'), \phi_\lambda^{(-)}(t)]|0\rangle\theta(t'-t). \end{aligned} \tag{6}$$

With the choice of $A^{(\pm)}$ given by equation (5) it is easy to see that this propagator is not invariant with respect to translations. In order to show this it is necessary to look at only one of the terms in equation (6). To second order in λ we have

$$\begin{aligned} &[\phi_\lambda^{(+)}(t), \phi_\lambda^{(-)}(t')] \\ &\quad \simeq [\phi_0^{(+)}(t), \phi_0^{(-)}(t')] + \lambda[\phi_1^{(+)}(t), \phi_0^{(-)}(t')] + \lambda[\phi_0^{(+)}(t), \phi_1^{(-)}(t')] \\ &\quad\quad + \lambda^2[\phi_1^{(+)}(t), \phi_1^{(-)}(t')] + \lambda^2[\phi_2^{(+)}(t), \phi_0^{(-)}(t')] + \lambda^2[\phi_0^{(+)}(t), \phi_2^{(-)}(t')], \end{aligned} \tag{7}$$

where $\phi_1^{(+)}$ is

$$\phi_1^{(+)}(t) = -cA_0^{(+)3}\,\mathrm{e}^{-3imt} + A_1^{(+)}\,\mathrm{e}^{-imt} \tag{8}$$

with

$$A_0^{(+)} = a \tag{9}$$

$$A_1^{(+)} = c\alpha_1 a^3. \tag{10}$$

The propagator will contain no first order terms, and the only second order term arises from the commutator of $\phi_1^{(+)}$ with $\phi_1^{(-)}$. Using equations (8)–(10) this is

$$\begin{aligned} &\langle 0|[\phi_1^{(+)}(t), \phi_1^{(-)}(t')]|0\rangle \\ &\quad = 6c^2[\exp\{-3im(t-t')\} - \alpha_1\{\exp(-3imt + imt') \\ &\quad\quad - \exp(-imt + 3imt')\} + \alpha_1^2 \exp\{-im(t-t')\}]. \end{aligned} \tag{11}$$

It is evident that the middle term in equation (11) is not invariant with respect to time translations.

Now, the commutation relations have been expressed in a way that allows us to find a solution at $t = 0$ which is independent of λ. Although the solutions given by equations (2)–(5) continue to satisfy the differential equation at $t = 0$, this method singles out $t = 0$ as a special time. If we start the scattering problem at $t = 0$ then we would not expect the propagator to be invariant under translations. However, even in this case the solutions do not reduce to free fields again at $t = \infty$. On the other hand, if we restate the commutator and require that the equal-time commutator be independent of λ for arbitrary times there are no solutions for which $A^{(\pm)}$ are independent of time.

However, if $A^{(\pm)}$ are time dependent, the functions $\phi^{(\pm)}$ are no longer solutions of the differential equation. Finally, if we continue to state the scattering problem as beginning at $t = -\infty$ and ending at $t = +\infty$, the solutions given here are unphysical since the propagator is not invariant with respect to displacements.

Thus, we obtain the following from these solutions. If we restate the scattering problem to extend from $t = 0$ to $t = \infty$, we can satisfy canonical commutation relations with the solutions given by equations (2)–(5) at $t = 0$. Subsequently, the equal-time commutators are time dependent and also depend on the interaction. Free field operators do not appear again at $t = \infty$.

3. Commutators unspecified

We now turn to a quantized field theory in four dimensions appropriate to the scattering problem extending from $t = -\infty$ to $t = \infty$. The field equations have solutions (see appendix 1)

$$\phi_{\lambda}^{(\pm)} = \psi^{(\pm)}(1+c\lambda\psi^{(\pm)2})^{-1}, \tag{12}$$

if

$$(\partial_{\mu}\partial^{\mu}+n^2m^2)\psi^{(\pm)n} = 0. \tag{13}$$

A pair of particular solutions of equations (12)–(13) is, for a system of volume v,

$$\psi_{\boldsymbol{k}}^{(\pm)} = a^{(\pm)}(\boldsymbol{k})\exp(\mp i\check{k}\,.\,\check{x})v^{-1/2} \tag{14}$$

with

$$\check{k}\,.\,\check{k} = k_{\mu}k^{\mu} = k_0^2-\boldsymbol{k}^2 = m^2. \tag{15}$$

Now, instead of requiring the commutator to be independent of λ as in the previous section, we will leave it unspecified and instead, assume $a^{(\pm)}(\boldsymbol{k})$ to be $a(\boldsymbol{k})$ and $a^{\dagger}(\boldsymbol{k})$ respectively of the linear theory. Their commutator is taken to be

$$[a(\boldsymbol{k}), a^{\dagger}(\boldsymbol{k}')] = \delta_{\boldsymbol{k},\boldsymbol{k}'}k_0. \tag{16}$$

Thus, we quantize the nonlinear theory by knowing that there are operator-valued solutions of equation (13) for which the coefficients satisfy the commutation relations given in equation (16).

Next, we consider the construction of the propagator corresponding to these solutions. We first review the steps in constructing a propagator in the linear theory.

The first step is to find the solutions of the linear field equation. These have the form given in equation (14). The physical interpretation given to the solutions in equation (14) is that the quantity $\langle\epsilon|a^{\dagger}(k)\exp(i\check{k}\,.\,\check{x})/\sqrt{v}|0\rangle$ is the conditional probability amplitude that a particle of momentum $\boldsymbol{k}$ is created from the vacuum at the point x and propagates to the state described by $\langle\epsilon|$ with momentum $\boldsymbol{k}$. The total amplitude for a particle to be created at x and propagate to the state described by $\langle\epsilon|$ is

$$P(\epsilon, x) = \sum_{\boldsymbol{k}} k_0^{-1}\langle\epsilon|\psi_{\boldsymbol{k}}^{(-)}|0\rangle. \tag{17}$$

Assuming that the set of states $\langle\epsilon|$ is complete, the total propagator is

$$P(y, x) = \sum_{\epsilon,\boldsymbol{k},\boldsymbol{q}} k_0^{-1}q_0^{-1}(\langle 0|\psi_{\boldsymbol{q}}^{(+)}(y)|\epsilon\rangle\langle\epsilon|\psi_{\boldsymbol{k}}^{(-)}(x)|0\rangle\theta(y_0-x_0) + \langle 0|\psi_{\boldsymbol{q}}^{(+)}(x)|\epsilon\rangle\langle\epsilon|\psi_{\boldsymbol{k}}^{(-)}(y)|0\rangle\theta(x_0-y_0)) = i\Delta_F(x-y). \tag{18}$$

We will construct the propagator for the nonlinear theory in the same way. That is, $\phi_{\boldsymbol{q}}^{(-)}(x)$ is interpreted as the operator which creates a system at the point x with quantum numbers $\boldsymbol{q}$. However, $\boldsymbol{q}$ is no longer the momentum of the system. Thus, we assume that the self-interacting field has substates which can be labelled by the momenta of the associated free field. With this assumption and the solutions given in equations (12) and (14)–(16) the propagator for the self-interacting field is

$$P(x, y) = \sum_{\boldsymbol{k},\boldsymbol{q}} k_0^{-1} q_0^{-1} (\langle 0|[\phi_{\boldsymbol{k}}^{(+)}(x), \phi_{\boldsymbol{q}}^{(-)}(y)]|0\rangle \theta(x_0 - y_0) + \langle 0|[\phi_{\boldsymbol{k}}^{(+)}(y), \phi_{\boldsymbol{q}}^{(-)}(x)]|0\rangle \theta(y_0 - x_0))$$

$$= \sum_{n=0}^{\infty} \sum_{\boldsymbol{k}} (2n+1)! k_0^{2n-1} (c\lambda/v)^{2n} [\exp\{-(2n+1)\mathrm{i}\check{k}\,.\,(\check{x}-\check{y})\}\theta(x_0 - y_0)$$

$$+ \exp\{+(2n+1)\mathrm{i}\check{k}\,.\,(\check{x}-\check{y})\}\theta(y_0 - x_0)]$$

$$= \sum_{n=0}^{\infty} \frac{(2n+1)!}{(2n+1)^{2n+2}} (c\lambda/v)^{2n} \{(2n+1)^2 m^2 - \nabla^2\}^n \Delta_{\mathrm{F}}(x-y\,;(2n+1)^2 m^2) \qquad (19)$$

where we have used the integral representation for the step function

$$\theta(\pm z) = \mp \frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{d}h}{h \pm \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}ahz} \qquad (a > 0) \qquad (20)$$

and also the identity

$$\sum_{\boldsymbol{k}} f(\boldsymbol{k}) k_0 \exp(\mp \mathrm{i}\check{k}\,.\,\check{x})$$

$$= \sum_{\boldsymbol{k}} f(\boldsymbol{k})(k^2 + m^2)^{1/2} \exp(\mp \mathrm{i}\check{k}\,.\,\check{x}) = (m^2 - \nabla^2)^{1/2} \sum_{\boldsymbol{k}} f(\boldsymbol{k}) \exp(\mp \mathrm{i}\check{k}\,.\,\check{x}). \qquad (21)$$

It is evident that the propagator given by equation (19) is invariant under translations. The convergence properties of the series are discussed in appendix 2.

Now, it is also apparent that the equal-time commutator of $\phi_{\boldsymbol{k}}^{(+)}(x)$ and $\phi_{\boldsymbol{q}}^{(-)}(y)$ will depend on the coupling constant, that is,

$$[\phi_{\boldsymbol{k}}^{(+)}(x), \phi_{\boldsymbol{q}}^{(-)}(y)] = \sum_{n,p=0}^{\infty} (c\lambda/v)^{n+p} \exp(-\mathrm{i}n\check{k}\,.\,\check{x} + \mathrm{i}p\check{q}\,.\,\check{y})[a(\boldsymbol{k})^n, a^\dagger(\boldsymbol{q})^p],$$

which, even for equal times contains λ. Furthermore, this quantity is an operator.

Since the function $\Delta_{\mathrm{F}}(x-y\,;(2n+1)^2 m^2)$ enters the complete propagator given in equation (19) it is clear that this has poles at $(2n+1)m$, where m is the mass of the associated free field and n is an integer. If we retain the interpretation that the propagator has poles at physical states, we conclude that the field contains particles of mass $(2n+1)m$. The contribution of these higher mass states is proportional to λ^{2n} and thus vanishes with λ.

4. Summary

Two examples of self-interacting quantum field theories have been constructed from fields which satisfy the interacting field equations for all times. In the first example, by requiring the equal-time commutators of the positive and negative frequency field operators to be independent of the coupling constant at $t = 0$ a set of operators is obtained which is appropriate for describing a scattering problem extending from $t = 0$ to $t = \infty$. The fields contain the coupling constant for large times. In the second example the solutions to the interacting field equations are constructed from solutions of the free field equations in a way which insures that the propagator will be invariant with respect to translations. These solutions also contain the coupling constant λ at large times and are appropriate to describe a scattering problem extending from $t = -\infty$ to $t = +\infty$. The equal-time commutators depend on λ.

In the second example the requirement of translational invariance on the propagator essentially determines the properties of the coefficients in the solutions of the field equations. The result is that the equal-time commutator of the positive and negative frequency operators can no longer be specified independently. While the propagator does not have the mathematical properties in the interacting theory that it does in the free field theory, that is, it is not a Green function of the field equation—in fact, it does not even satisfy the field equation—its physical role remains the same. It is used to calculate transition matrix elements and consequently, is closely related to observable quantities. Therefore, the requirement of translational invariance seems well motivated.

Further, in the second example, the propagator has poles at $(2n+1)m$, so the theory describes systems composed of more than one particle. (However, the position of the poles is independent of the coupling constant. This is similar to the behaviour of the scattering amplitude when a new multiparticle channel is opened. It is possible that the particular solutions of the field equations used in calculating the propagator distort the general solution, with the net effect of the distortion being that a branch cut in the propagator appears as a pole.) This complexity in the field is also reasonable if we expect the field to reflect the complexity of observed physical systems such as hadrons.

Finally, the point of view adopted here has emphasized the role of the field equations and their solutions independent of perturbation theory, rather than the existence of the energy–momentum tensor. Consequently, the connection between this point of view and the canonical formalism, with its emphasis on the hamiltonian, is obscure. This question remains to be investigated, although with the explicit solutions given here we expect that a better understanding of these two points of view is possible. Another question which can be examined directly with these explicit solutions is the assumption that the bare mass of a particle can be corrected to its physical value due to a self-interaction. These, and other questions, will be considered subsequently.

Appendix 1

In this appendix we consider the details of the solutions to equation (1) given in equations (12)–(13). This differential equation is a special case of some nonlinear equations discussed elsewhere (Reid and Burt 1973). The solutions are constructed in terms of solutions of the related linear differential equation (called the base equation)

$$(\partial_\mu \partial^\mu + m^2)\psi = 0. \tag{A1.1}$$

As is well known, this equation has both c number and operator solutions which may be characterized as positive or negative frequency solutions, for example,

$$\psi^{(\pm)} = \int d^4k \delta(k^2 - m^2) \exp(\mp i\check{k} . \check{x}) A^{(\pm)}(\boldsymbol{k}). \tag{A1.2}$$

Two particular solutions of equation (1), constructed from either the positive or negative frequency solutions of (A1.1), are

$$\phi_\lambda^{(\pm)} = \psi^{(\pm)}(1+c\lambda\psi^{(\pm)^2})^{-1}, \tag{A1.3}$$

provided

$$\partial_\mu\psi^{(\pm)}\partial^\mu\psi^{(\pm)}+m^2\psi^{(\pm)^2} = 0, \tag{A1.4}$$

$$[A^{(+)}(\boldsymbol{k}), A^{(+)}(\boldsymbol{k}')] = [A^{(-)}(\boldsymbol{k}), A^{(-)}(\boldsymbol{k}')] = 0, \tag{A1.5}$$

and

$$c = -\frac{1}{8m^2}. \tag{A1.6}$$

Condition (A1.5) is the condition applied to operator solutions of (A1.1) in quantum field theory.

These results may be verified by direct substitution. Differentiating (A1.3), using (A1.1) and (A1.4)–(A1.6) we find

$$\begin{aligned}\partial_\mu\phi_\lambda^{(\pm)} &= \partial_\mu\psi^{(\pm)}(1+c\lambda\psi^{(\pm)^2})^{-1}-2c\lambda\partial_\mu\psi^{(\pm)}\psi^{(\pm)^2}(1+c\lambda\psi^{(\pm)^2})^{-2}\\ &= -\partial_\mu\psi^{(\pm)}(1+c\lambda\psi^{(\pm)^2})^{-1}+2\partial_\mu\psi^{(\pm)}(1+c\lambda\psi^{(\pm)^2})^{-2}\end{aligned} \tag{A1.7}$$

and

$$\begin{aligned}\partial_\mu\partial^\mu\phi_\lambda^{(\pm)} &= -\partial_\mu\partial^\mu\psi^{(\pm)}(1+c\lambda\psi^{(\pm)^2})^{-1}+2c\lambda\partial_\mu\psi^{(\pm)}\partial^\mu\psi^{(\pm)}\psi^{(\pm)}(1+c\lambda\psi^{(\pm)})^{-2}\\ &\quad+2\partial_\mu\partial^\mu\psi^{(\pm)}(1+c\lambda\psi^{(\pm)^2})^{-2}-8c\lambda\partial_\mu\psi^{(\pm)}\partial^\mu\psi^{(\pm)}\psi^{(\pm)}(1+c\lambda\psi^{(\pm)^2})^{-3}\\ &= -m^2\phi_\lambda^{(\pm)}-\lambda\phi_\lambda^{(\pm)^3}.\end{aligned} \tag{A1.8}$$

Clearly, rearranging (A1.8) gives equation (1).

It is also easy to see that (A1.4) leads to (13) directly. We have using (A1.1)

$$\partial_\mu\partial^\mu\psi^{(\pm)n} = (n-1)n\partial_\mu\psi^{(\pm)}\partial^\mu\psi^{(\pm)}\psi^{(\pm)n-2}+n\partial_\mu\partial^\mu\psi^{(\pm)}\psi^{(\pm)n-1} = -n^2m^2\psi^{(\pm)n}, \tag{A1.9}$$

for any real n.

The remaining question is to show that the forms given in (A1.3)–(A1.6) are particular solutions and not singular solutions. This proof is elementary for $\phi_\lambda^{(\pm)}$ which are functions of $k_0x_0-\boldsymbol{k}\cdot\boldsymbol{x}$ and will not be given here (see Reid and Burt 1973).

Appendix 2

The functions appearing in the propagator given in equation (10) are the usual singular functions appearing in quantum field theory. However, we must also consider separately the convergence of the series in equation (10). For this purpose we will use explicit representations of the function $\Delta_F(x; m^2)$ (Bjorken and Drell 1965 appendix C). These are

$$\Delta_F(x) = -\tfrac{1}{2}\mathrm{i}\Delta_1(x)+\tfrac{1}{2}\epsilon(x_0)\Delta(x), \tag{A2.1}$$

where

$$\epsilon(z) = \begin{cases} 1 & z>0\\ -1 & z<0.\end{cases} \tag{A2.2}$$

The functions Δ_1 and Δ may be written

$$\Delta(x) = \frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} J_0(m(t^2-r^2)^{1/2}) & t > r \\ 0 & -r < t < r \\ -J_0(m(t^2-r^2)^{1/2}) & t < -r \end{cases} \tag{A2.3}$$

and

$$\Delta_1(x) = \frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} Y_0(m(t^2-r^2)^{1/2}) & |t| > r \\ -(2/\pi)K_0(m(r^2-t^2)^{1/2}) & |t| < r \end{cases} \tag{A2.4}$$

where J_0, Y_0 and K_0 are cylinder functions (Watson 1944). From these results it is clear that for time-like intervals one has

$$\{(2n+1)^2m^2-\nabla^2\}^n \mathrm{i}\Delta_{\mathrm{F}}(x;(2n+1)^2m^2)$$

$$= \{(2n+1)^2m^2-\nabla^2\}^n(8\pi r)^{-1}(\partial/\partial r)H_0^{(2)}((2n+1)m(t^2-r^2)^{1/2}) = \mathscr{P}_n \tag{A2.5}$$

where $H^{(2)}$ is the second Hankel function, and for space-like intervals

$$\mathscr{P}_n = \{(2n+1)^2m^2-\nabla^2\}^n(8\pi r)^{-1}(\partial/\partial r)K_0((2n+1)m(r^2-t^2)^{1/2}). \tag{A2.6}$$

Using these results the convergence properties of the series in equation (10) can be readily established. We will illustrate the argument for space-like intervals.

First, define σ as

$$\sigma = (r^2-t^2)^{1/2}. \tag{A2.7}$$

Then, the nth term in the series is $(P_n = \{(2n)!/(2n+1)^{2n+1}\}(c\lambda/v)^{2n}\mathscr{P}_n)$

$$P_n = \frac{(2n)!}{(2n+1)^{2n+1}}\left(\frac{c\lambda}{v}\right)^{2n}\left\{(2n+1)^2m^2-\frac{\partial^2\sigma}{\partial r^2}\frac{\mathrm{d}}{\mathrm{d}\sigma}-\left(\frac{\partial\sigma}{\partial r}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}-\frac{2}{r}\frac{\partial\sigma}{\partial r}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right\}^n$$

$$\times(2\pi^2)^{-1}\left(\frac{\partial}{\partial r}\right)K_0((2n+1)m\sigma) \tag{A2.8}$$

where we have used (A2.7) and the result that there is no angular dependence in K_0. In order to avoid the light cone singularity in P_n we will examine this expression at points far from the light cone. A more complete argument will be given elsewhere (Burt 1973). For large σ we have

$$K_0((2n+1)m\sigma) \simeq [\pi/\{2(2n+1)m\sigma\}]^{1/2}\exp\{-(2n+1)m\sigma\}. \tag{A2.9}$$

Performing the differentiations indicated in (A2.8) we find, for large n, that the differential operator may be replaced by the first and third terms, giving the approximate result

$$P_n \simeq \frac{(2n)!}{(2n+1)^{1/2}}\left(\frac{m\pi}{2\sigma^3}\right)^{1/2}(-z^2)^n\,\mathrm{e}^{-m\sigma} \tag{A2.10}$$

where

$$z = \frac{cm\lambda}{v}\frac{|t|}{\sigma}\mathrm{e}^{-m\sigma}. \tag{A2.11}$$

Now, it is evident that this series diverges rapidly for fixed z. Furthermore, the source of the divergence is apparent. The presence of the term $\langle 0|[a^n(\boldsymbol{k}), a^{n'}(\boldsymbol{k}')|0\rangle$ in the series is the scalar product of two states, $a^{\dagger n'}(\boldsymbol{k})|0\rangle$ and $\langle 0|a^n(\boldsymbol{k})$ without the normalization factors $(n'!)^{-1/2}$ and $(n!)^{-1/2}$. In the linear theory we are free to insert these factors in the construction of the Hilbert space. However, in the nonlinear theory these operators arise from the expansion of the operator $(1+c\lambda A^2)^{-1}$ in powers of λ and consequently contain the numerical coefficients appropriate to this series.

References

Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (New York: McGraw-Hill)
Burt P B 1973 *Phys. Lett.* B submitted for publication
Burt P B and Reid J L 1972 *J. Phys.* A: *Gen. Phys.* **5** L88
Glimm J and Jaffe A 1970 *Quantum Field Theory Models* (Les Houches Summer School 1970) (New York: Gordon and Breach)
Matthews P T 1950 *Phil. Mag.* **41** 185
Reid J L and Burt P B 1973 *J. math. Anal. Appl.* **44** in the press
Segal I E 1960 *J. math. Phys.* **2** 468
—— 1964 *J. math. Phys.* **5** 269
Streater R F and Wightman A S 1964 *PCT, Spin and Statistics and All That* (New York: Benjamin)
Watson G N 1944 *A Treatise on the Theory of Bessel Functions* (Cambridge: Cambridge University Press)